A MATHEMATICAL MODEL OF THE 100M AND WHAT IT MEANS

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Summary

This paper develops a mathematical model, or formula, of the 100m sprint, which accurately describes the relationship between time elapsed and distance covered in the race. From this are derived formulae for velocity and acceleration. There are four constants in these formulae. They are constant throughout a particular performance of a particular runner, and they vary from performance to performance and runner to runner. The constants have important significance, and describe the maximum speed to which the runner is heading, the acceleration, the extent of loss of speed at the end, and the moment of onset of deceleration. This enables performance to be compared with performance.

The model is applied to the men's 100m event at the 1999 World Championships, for which times at each 10m-point are available for all finalists. From this set of data models are derived for the average performance of the first seven runners, representing archetypal elite sprinter, and for the winner, Maurice Greene. The differences between Greene and the average finalist are identified. The really significant result to come out of the analysis is that Greene is able to accelerate for considerably longer than anyone else.

Introduction

It is common in Mathematics, Science, Engineering, Economics and other quantitative disciplines to develop models that describe and predict behaviour of systems. A simple example is the model a car manufacturer might develop to describe the relationship between speed and fuel consumption of his vehicle. Running is an activity that is governed by certain dynamics, which are amenable to mathematical description. Furthermore the output, namely the performance, can be quantified, in terms of time, distance, speed and acceleration. This should enable mathematical models to be developed that can describe the effect of various factors on performance, and possibly even can predict performance. In particular the 100m event is devoid of unpredictable variations like tactics, and is an all out effort from start to finish, so it is ideal for mathematical modelling.

Lames (1990) pointed out that there are two approaches to modelling of the 100m. Both make use of the output of a performance- namely the elapsed times at regular intervals of the race, normally every 10 metres. The first and the simpler approach is to find a function that reasonably fits the output data. The second asks what function describes the functioning of the runner, and then fits this function to the output data. Lames preferred the second and the author agrees. The 100m performance is characterised by two distinct parts- acceleration and deceleration. There are exponential functions that describe these well, and can be combined to describe the whole 100m. Way back in 1951 according to Lames, Henry and Tafton used the acceleration function for the 60 yard dash. Lames, at the VIII International Symposium of the Society of Biomechanics in Sport in Prague in 1990 introduced a model based on the acceleration and deceleration functions, and it is this that the author develops in this paper.

What is a Model of 100m?

What we are doing when we model a 100m performance of a particular runner is to obtain functions *s(t)*, *v(t)* and *a(t)* of distance, velocity and acceleration in terms of time, which will enable us to say where he was, how fast he was going, and how much he was accelerating, at any stage during the performance. The independent variable is time (*t*), and the dependent variables are distance, velocity and acceleration. The functions are functions of *t*, and they have in them a set of constants. It is the same general function for all performances, but the constants differ from performance to performance. In other words the constants are the unknowns, and we use the distance and time data from a particular performance to determine the constants for that performance. That gives us a definite set of functions for the performance that enable us to calculate distance, velocity and acceleration at any time during the performance. Then we can compare performance with performance, and we have solid information that enables us to say what made a performance good or not so good.

Discussion of Modelling of 100m

The information that is usually available about a 100m performance is the elapsed time at the end of each 10m section. The obvious method of modelling is to calculate the times for each 10m section from this data, and then calculate the average speeds for each 10m. This in turn is regarded as a set of points on the velocity-time or velocity-distance curve, and the equation of the curve is determined. There are three problems with this approach. In the first place the real velocity-time or velocity-distance curve is a continuous curve of instantaneous velocity, and the speeds that are derived from the section times are average speeds for the sections. They do not lie on the real curve, so a model derived from them is misleading. Secondly they are based on differences between consecutive elapsed times, and any errors in the elapsed times are magnified in the differences. In other words the section times and hence the calculated speeds are not sufficiently accurate. A time that is calculated as 0.85 might really be closer to 0.86, and this could make a difference in deciding whether a runner is accelerating or decelerating. Thirdly, the maximum speed indicated by the section times is not the real maximum velocity, only the maximum average speed. Furthermore the real maximum might not even lie in the section of maximum average speed. It could have been in the section before or after. So again it is misleading.

An indication of the error possible in section times is illustrated in the IAAF times in the Men's 100m at the 1991 World Championships. All of the finalists are shown as reaching their fastest time in the 70-80m section, even thought most of them were shown as being in deceleration before that. That is a most unlikely occurrence and hints at a possible error in location of one or more 10m marks. Even if it were true it could only have been due to an influence outside the runners. A model derived from this data is likely to misleading, and information derived from it will not be reliable.

In order to avoid the errors associated with the differencing, it is preferable to use the raw data of elapsed times at the 10m interval distances. This necessitates a distance-time relationship rather than velocity-time or velocity-distance. The problem with this is that distance and time are outputs of the run and not underlying processes. On the other hand acceleration and velocity are underlying processes, which produce the output. They are something a runner does in order to reach points along the track in certain times. As remarked above, there are good models for them that do describe what the runner is doing. Since these are continuous functions, we can use calculus to derive from them a continuous distance-time function. We then use the raw data to determine the constants of this function for a particular performance, and hence also of the velocity and acceleration functions of that performance. We then have a complete description of the performance that enables us to analyse it in great detail.

Another advantage of obtaining the model in the above manner is that errors in the elapsed times are smoothed by techniques that produce the constants of the distance-time relationship. The techniques find a continuous relationship, which mirrors the continuity of the run, so any discontinuities resulting from errors in the elapsed times are ironed out.

The Model

The sprinter applies maximum power to propel himself from a stationary position to maximum velocity as soon as possible. Maximum force is applied initially, and since $F=ma$, the initial acceleration is maximum. The early velocity is slow but builds rapidly due to the large acceleration. Soon the acceleration diminishes and the velocity gains reduce. The acceleration falls away to nothing, and the velocity becomes somewhat flat. Then the acceleration becomes negative and the velocity falls for the last part of the performance.

There is a very good mathematical function that describes the first part of the performance, in terms of velocity, up to the point prior to velocity beginning to fall. It is an exponential function, i.e. it contains the number e (approximately 2.718), much beloved of mathematicians and scientists, which is used in describing growth or decay. The basis of the function is

$$f(t) = 1 - e^{-kt}$$

and a graph of a typical function in the range 0 to 10 seconds is shown below.

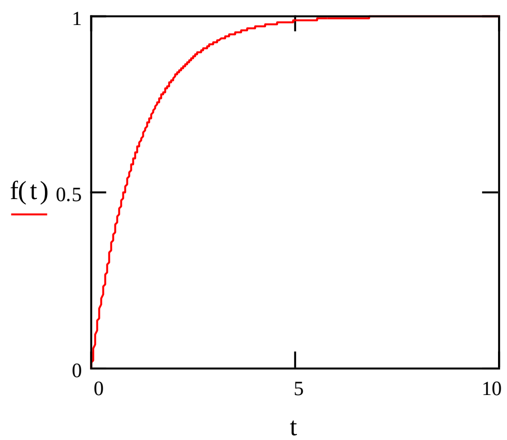

Figure 1. Example of decreasing growth exponential function

The second and last part of the 100m is one of slight deceleration. Again there is an exponential function that models this. It begins with a slight descent at *t=0*, and the descent accelerates rapidly as t increases. It is the first part, the slight descent, that is used to model the end of the 100m. The function is

$$f(t) = 1 - e^{lt}$$

and a graph for a typical function in the range 0 to 10 seconds is shown below.

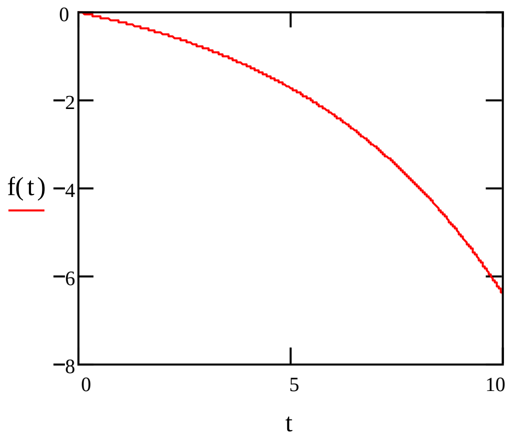

Figure 2. Example of increasing decay exponential function

Of course the fall of in velocity is not as great as indicated in the graph, but this is handled by a multiplier in front of the function. However it is the right shape.

Following Lames we combine the above two functions to obtain the model for the full 100m. It is

$$v(t) = A(1 - e^{-kt}) + F(1 - e^{lt})$$

The constants in this equation are A, k, F and l. They each have special significance, and this will be explained as we go on. The constants are such that the second part of the model is very small for the first half of the race, and has insignificant impact on the first part of the model. It is only into the second half that the size of the second part becomes significant, and begins to result in an overall fall in the value of v. The graph of a typical model is shown below.

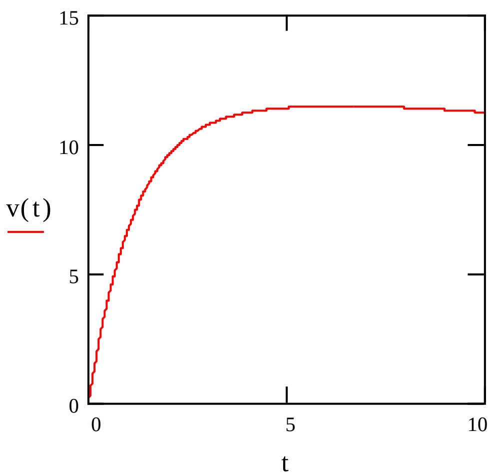

Figure 3 Graph of a velocity model of a typical 100m

We obtain the distance part of the model by integrating the velocity function. This gives us

$$s = At + (A/k).e^{-kt} + Ft + (F/l).e^{lt} + c$$ (where c is the constant of integration)

When t=0, s=0

therefore $0 = A/k - F/l + c$

so $c = F/l - A/k$

and $s = (A+F)t - (A/k).(1 - e^{-kt}) + (F/l).(1 - e^{lt})$.

The acceleration is obtained by differentiating the velocity function.

$$a = Ake^{-kt} - Fle^{lt}.$$

The Significance of A, k, F and l

A is the limit which the velocity of the runner approaches as he attains top speed. He never reaches *A*, partly because it is only a limit to which the velocity tends, and also because the second part of the function, the deceleration part, begins to be significant enough to reduce the speed. Nevertheless it is a measure of the absolute speed of a runner.

Some more calculus, which the author will not produce here, shows that for *k* less than 1.0, the larger *k* the larger will be the acceleration. *k* will be less than 1.0 for normal 100m races.

F is a velocity reducer. The larger *F* the greater will be the fall in velocity at the end of the race.

l is the main determiner of the instant deceleration begins. Deceleration begins when acceleration is zero, and calculus shows us that this instant is given by

$$t = (\log_e Ak - \log_e Fl)/(l + k)$$

The smaller *l* the larger will be the value of *t*.

Use of the Model- 1999 World Championships 100m

CSD (1999) published elapsed times at 10m intervals for the finalists in the 1999 World Championships Men's 100m. From these the author obtained two models. The first was a profile of the performance of an elite sprinter. The second was a profile of the winner, Maurice Greene.

Lames had a problem with his model, which he related in his 1990 paper. It was in obtaining a value for *l*. The problem was instability in the value derived for *l*, and he said this arose because of the few input points (11) and the fact that really only the last few points had much bearing on the value. Consequently he had to choose a value for *l* rather than calculate it, which left a little doubt about the model. The author had better fortune, probably due to having a better computer program. The program is known as "genfit" and is in a Microsoft software package known as "Mathcad". The program is quite sophisticated. It requires the assumed basic function to be input, together with a guess at the constants. Its sophistication is in the use of partial derivatives of the function with respect to the constants. Furthermore the author used the program iteratively, continuing to take the constants obtained from the program and input them as the guess, until output constants were the same as the input. This provided all the stability required. A worksheet of the program is included at Appendix 1.

Another necessary refinement is to work with actual running times rather than times from the gun. This is because the model is modelling actual running, and not the reaction time. So the reaction time was subtracted from the elapsed time before the data was fed into the program. Taking this into account, the distance function becomes

$$s = (A + F)(t-r) - (A/k).(1 - e^{-k(t-r)}) + (F/l).(1 - e^{l(t-r)}),$$ where r is reaction time.

In the use of the data from the 1999 World Championships, the data from the eighth runner was omitted. The first seven runners ranged from 9.80 to 10.07 seconds, while the eighth recorded 10.24 seconds. The latter for some reason obviously did not belong with the others on that occasion, so inclusion would distort the picture of the elite sprinter.

The times of the first seven runners were averaged at each 10m point, in order to obtain a profile of a typical elite sprinter. The average reaction time was 0.139 seconds, so 0.14 was subtracted from the average of the elapsed times at each 10m mark. These were put into the program genfit as the *x*-vector, together with the 10m marks as the *y*-vector. A guess of the values of the constants was also input. The output from the program was the calculated values of the constants. These were fed back in as the guess (now an estimate), and this process continued until the output was the same as the estimate. The output then was the constants of the model. The model so obtained for the typical elite sprinter is

$$s = 11.707(t - 0.14) - 13.568(1 - e^{-0.8609(t - 0.14)}) + 0.0906(1 - e^{0.2848(t - 0.14)})$$

$$v = 11.707 - 11.681e^{-0.8609(t - 0.14)} - 0.0258e^{0.2848(t - 0.14)}$$

$$a = 10.056e^{-0.8609(t - 0.14)} - 0.007348e^{0.2848(t - 0.14)}$$

Proof that the model is a good one is the values of *s* that are obtained by putting into the model the average elapsed times at the 10m marks. This is shown in Appendix 2. Most of the distances are within a few centimetres of the actual distances, and three are closer than 1 centimetre. The only mild discrepancy is at 10m, where it is 13 cm, or 1.3%. Obviously the model is not as good for the first few metres, but this is only a minor concern.

The following comes out of the model for the typical elite sprinter:

Reaction time	0.14sec
Speed limit	11.68 m/s
Initial acceleration	10.05 m/s^2
Acceleration constant	0.8609
Duration of acceleration	6.44sec
Duration of deceleration	3.38sec
Point of max speed	59.79m
Max speed	11.50m/s
Total time	9.96sec

The same exercise was conducted on the data for Greene. His model was:

$$s = 11.777(t-0.13) - 13.684(1 - e^{-0.8600(t - 0.13)}) + 0.05325(1 - e^{0.1690(t - 0.13)})$$

$$v = 11.777 - 11.768e^{-0.8600(t - 0.13)} - 0.00900e^{0.1690(t - 0.13)}$$

$$a = 10.12e^{-0.8600(t - 0.13)} - 0.001521e^{0.1690(t - 0.13)}$$

The following comes out of the model for Greene:

Reaction time	0.13sec
Speed limit	11.77m/s
Initial acceleration	10.12m/s^2
Acceleration constant	0.8600
Duration of acceleration	8.68sec
Duration of deceleration	0.99sec
Point of max speed	86.84m
Max speed	11.73m/s
Total time	9.80sec

Implications

In many respects Greene is the typical elite sprinter, i.e. he is about the average of the first seven. His reaction time is a little quicker than average, but not significantly so. His initial acceleration is a little higher but his acceleration constant is a little lower. However the really significant difference is that he accelerates for appreciably longer. That is what gives him his superiority. While the others have reached their maximum speed and are into deceleration he is still getting faster. For 2 seconds he is getting faster while the rest are getting slower.

The practical lesson from this model for sprinters and coaches would seem to be the benefit of extending the time of acceleration. It is this, rather than raw power out of the blocks, that will result in faster times. It is probably a matter of control.

Conclusion

It is possible to derive a mathematical model that models a 100m performance very well. It provides valuable information on the makeup of the performance, in terms of acceleration, velocity and distance at any stage in the race. It enables us to see the vital ingredients of success in 100m running, and that the most vital is to accelerate as long as possible.

References

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